On Hilbert's Article, "On The Infinite"

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In his article, "On The Infinite," Hilbert begins with a celebration of Weierstrauss' work in the rigorization of the calculus. However despite calling it "a symphony of the infinite," he claims that the nature of the infinite was not fully investigated within its study. The key insight about infinity, he claims is provided by Cantor's work. There were key results about infinite sets that Cantor proved such as the fact that the naturals and rationals can be enumerated but the reals and naturals cannot. He writes that mathematicians were excited to work within in the realm of Cantor's Paradise that they did not pay close attention to the methods that they applied. It soon became apparent that new methods were needed if we were learn about the true nature of infinity.

But what is that? What is the "true" nature of infinity? The infinitely small does not seem to exist for we have the atom, the electron, and the "quanta." The infinitely large is a dubious claim because of our ever growing understanding of unbounded and finite geometries like elliptic or spherical.

Hilbert claims it is not impossible to investigate the true nature of infinity (whatever it may be) but we must tread lightly. Our reasoning may fool us if we devise arbitrary definitions, that is, if we blindly use logical rules without any motivation. He, then, reasons that mathematics that mathematics cannot be grounded solely on logic. (Then wouldn't Hilbert's program also have been doomed from the start?)

He then begins a crucial example to illustrate his point on the infinite. He states the theorem: $\exists p \in \mathbb{N} \cap [n+1, n+1!]$ such that p is prime. This statement fits finely within our finite universe because it is reducible to a finite list of statements, namely: "$n+1$ is prime or $n+2$ is prime or $n+3$ is prime $\ldots n!+1$ is prime." However if we take the weaker statement: $\exists p \in \mathbb{N} \cap [n+1, \infty]$, such that p is prime. We observe that this statement is not reducible to a finite list of existential statements. To prove this we may want to show that the negation of this statement is false and then apply the law of the excluded middle. In other worlds,we wish to derive a contradiction, as Hilbert has famously done with the Gordon Problem.

Hilbert transitions to the idea that we must supplement finite statements with "ideal" statements. What does he mean by this? Hilbert gives an example with commutative addition. In other words, $a + b = b + a$. "ideally" holds for ever object a and b. This is consistent with Hilbert's own philosophy of math where it such objects follow the rules of addition, whatever we deem it to be.