2 Analysis Proofs

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Theorem 1. *Let S be a nonempty subset of $\mathbb{R}$ that is bounded below, then S has an infimum.*

Proof. Let $T = \{b \in \mathbb{R} : b \leq a, \forall a \in S\}$, whose existence is guaranteed by the condition that S is a non-empty subset of $\mathbb{R}$ which is bounded below. By these same conditions, we also know that T is bounded above. Then, we apply the completeness property of $\mathbb{R}$ to obtain the fact that T has a supremum, which we may call m. Now we arbitrarily choose $y \in S$. We, then, know that y is an upper bound for T and, since m is the supremum of T we know that $m \leq y$. We then, know that m is a lower bound for S. Since m is still an upper bound for T, we know that every $x \in T$ has the property that $x \leq m$. We now that m is the infimum for S by the facts that for any $y \in S, m \leq y$ and for any $x \in T, x \leq m$. □

Theorem 2. *Given any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$ Similarly, there is an $l \in \mathbb{N}$ such that $-l < x$.*

Proof. Suppose the contrary, namely that there exists an $x \in R$ such that there is no $n \in \mathbb{N}$ for which the inequality $x < n$ is true. Hence, we have that for all $n \in \mathbb{N}$, with $n \leq x$. Therefore, x is an upper bound for $\mathbb{N}$ Hence, by the Completeness Property, we have that $\mathbb{N}$ has a supremum, which we may call M. We then know that $M - 1$ is not an upper bound for $\mathbb{N}$. Therefore, there exists some $n_0 \in \mathbb{N}$ such that $M - 1 < n_0$. But since $n_0 + 1$ is also an element of $\mathbb{N}$, we know that $M < n_0 + 1$. Hence, M is not an upper bound. This contradiction establishes the theorem. □

Corollary 2.1. *For any $x \in \mathbb{R}$, we have that there exists $m, n \in \mathbb{N}$ such that $x \in (-m, n)$.*