A Curious Case of $\sqrt{2}$ in $\mathbb{Q}$

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This is my annotation of the 2 pages of Rudin's "Principles of Mathematical Analysis." It is meant as an undergraduate introduction to mathematical analysis (hence the title). However, I find that is a very difficult book to read without being active. So this post is meant to show my thought process when absorbing material.

We begin our discussion of analysis with discussion of the rational numbers often denoted as $\mathbb{Q}$. Which is precisely the set $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}/\{0\}$. We quickly see that this insufficient to do analysis as it is not complete. Meaning if we map elements of $\mathbb{Q}$ unto a line by the function that maps a number to its corresponding length if we fix some unit distance, then we see that there are gaps in the line. This is easily seen by the fact there is no rational p that satisfies the equation $p^2 = 2$. But proof of this fact is required.

Theorem 1. *There is no rational p that satisfies the equation $p^2 = 2$.*

Proof. Suppose for the sake of contradiction that there is a rational p that satisfies the equation $p^2 = 2$. Then it follows that

$$
\begin{aligned}
(\frac{m}{n})^2 &= 2 \textit{ By rewriting } p \textit{ as a fraction} && (1)\\
\frac{m^2}{n^2} &= 2 \textit{ Distributing the square function} && (2)\\
m^2 &= 2n^2 \textit{ Multiplying both sides by } n^2 && (3)\\
m^2 &\in 2\mathbb{Z} \textit{ Since } m^2 \textit{ can be written as } 2 \textit{ times an integer} && (4)\\
m &\in 2\mathbb{Z} \textit{ Since the square function is closed in } 2\mathbb{Z} && (5)\\
m &= 2r \textit{ for some } r \in \mathbb{Z} \textit{ by the above statement} && (6)\\
4r^2 &= 2n^2 \textit{ By substituting the new term for } m && (7)\\
2r^2 &= n^2 \textit{ By dividing by } 2 && (8)\\
n^2 &\in 2\mathbb{Z} \textit{ Since } n^2 \textit{ can be written as } 2 \textit{ times an integer} && (9)
\end{aligned}
$$

$$n \in 2\mathbb{Z} \text{ Since the square function is closed in } 2\mathbb{Z} \tag{10}$$

The fact that $n, m \in 2\mathbb{Z}$ is the desired contradiction since assuming here is a rational p that satisfies the equation $p^2 = 2$ leads to the existence of an irreducible fraction that has a factor of 2 in both its numerator and denominator. □

Let's examine the curious case of $p^2 = 2$. Consider a subset A of $\mathbb{Q}^+$ (the positive rationals) that satisfy the equation $a^2 < 2$. And the subset B of $\mathbb{Q}^+$ that satisfying the equation $b^2 > 2$. We intuitively may think of A and B as being both sides of the line defined earlier separated by $\sqrt{2}$ (whatever that means). We, then, have the following theorem.

Theorem 2. *A has no greatest element and B has no smallest. More explicitly, for each $p \in A$, there exists $q \in A$ such that $p < q$. Similarly, for each $p \in B$, there exists $q \in B$ such that $q < p$.*

Proof. We fix p. It does not matter whether p belongs to A or B. Then, we define q as described in statement of the theorem as

$$q = p - \frac{p^2 - 2}{p + 2}$$

Note that $p + 2$ is non-zero because of the condition that $p \in \mathbb{Q}^+$

$$q = \frac{p(p+2)}{p+2} - \frac{p^2-2}{p+2} \text{ Multiplying } p \text{ by } [\tfrac{p+2}{p+2} = 1] \tag{11}$$

$$q = \frac{p^2+2p}{p+2} - \frac{p^2-2}{p+2} \text{ Cleaning up the numerator} \tag{12}$$

$$q = \frac{p^2+2p-(p^2-2)}{p+2} \text{ Subtracting the elements of the RHS} \tag{13}$$

$$q = \frac{2p+2}{p+2} \text{ Cleaning up the numerator} \tag{14}$$

$$q^2 = (\frac{2p+2}{p+2})^2 \text{ Squaring both sides} \tag{15}$$

$$q^2 - 2 = (\frac{2p+2}{p+2})^2 - 2 \text{ Subtracting both sides by 2} \tag{16}$$

$$q^2 - 2 = \frac{(2p+2)^2}{(p+2)^2} - 2 \text{ Distributing the square} \tag{17}$$

$$q^2 - 2 = \frac{(2p+2)^2}{(p+2)^2} - 2\frac{(p+2)^2}{(p+2)^2} \text{ Multiplying 2 by a special 1.} \tag{18}$$

$$q^2 - 2 = \frac{4p^2+8p+4}{(p+2)^2} + \frac{-2p^2-8p-8}{(p+2)^2} \textit{ Expanding RHS} \tag{19}$$

$$q^2 - 2 = \frac{2p^2-4}{(p+2)^2} \textit{ Combining like terms} \tag{20}$$

$$q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2} \textit{ Desired equation} \tag{21}$$

Suppose $p \in A$, then by definition $p^2 < 2$

$$p^2 - 2 < 0 \textit{ By hypothesis} \tag{22}$$

$$\frac{p^2-2}{p+2} < 0 \textit{ Since } p \in \mathbb{Q}^+ \tag{23}$$

$$-\frac{p^2-2}{p+2} > 0 \textit{ Multiplying both sides by } -1 \tag{24}$$

$$p - \frac{p^2-2}{p+2} > p \textit{ Adding p to both sides.} \tag{25}$$

$$q > p \textit{ LHS is defined to be the expression for q} \tag{26}$$

$$p^2 - 2 < 0 \textit{ Rewriting hypothesis} \tag{27}$$

$$\frac{2(p^2-2)}{(p+2)^2} < 0 \textit{ Multiplying by positive factor} \tag{28}$$

$$q^2 - 2 < 0 \textit{ Substituting LHS} \tag{29}$$

The results that $q > p$ and $q^2 - 2 < 0$ establish the first part of the theorem. Now for the 2^{nd} part, which is done in a similar manner.

$$p^2 - 2 > 0 \textit{ By hypothesis} \tag{30}$$

$$\frac{p^2-2}{p+2} > 0 \textit{ Since } p \in \mathbb{Q}^+ \tag{31}$$

$$-\frac{p^2-2}{p+2} < 0 \textit{ Multiplying both sides by } -1 \tag{32}$$

$$p - \frac{p^2-2}{p+2} < p \textit{ Adding p to both sides} \tag{33}$$

$$q < p \textit{ LHS is defined to be the expression for q} \tag{34}$$

$$p^2 - 2 > 0 \textit{ Rewriting hypothesis} \tag{35}$$

$$\frac{2(p^2-2)}{(p+2)^2} > 0 \textit{ Multiplying by positive factor} \tag{36}$$

$$q^2 - 2 > 0 \textit{ Substituting LHS} \tag{37}$$

The results that $q < p$ and $q^2 - 2 > 0$ establish the second part of the theorem. We are done. $\square$