Relations

Matthew Hanna

April 23, 2019

There are some particularly important relations to consider when doing math. Let R be a relation on X, that is $R \subset X \times X$. We say R is reflexive if xRx for each $x \in X$, that is for all $x \in X$, we have $(x,x) \in R$. Standard equality has this relation because obviously if $x = x$ then they are the same object. The inclusion relation has this property as well because every set is a subset of itself. A relation is said to be reflexive if whenever xRy ,we have yRx. In standard language, we have $[(x,y) \in R] \rightarrow [(y,x) \in R]$. Equality obviously satisfies this. However inclusion, in general, does not. We say that R is transitive if whenever xRy and yRz hold, we have xRz. In formal language we have $[(x,y) \in R \wedge (y,z) \in R] \rightarrow (x,z) \in R$. Equality and inclusion both satisfy this. If a relation R has the properties of symmetry, reflexivity, and transitivity, we say R is an equivalence relation.

Let X be any set. Then let $\{X_{\alpha \in A}\}$ be a family of subsets indexed by A with the properties that $\forall i, j \in A, X_i \cap X_j = \emptyset$ and $\cup_{\alpha \in A}\{X_\alpha\} = X$, we say that $\{X_\alpha\}$ partition the set X. We call each X_α a cell. It can be shown that with any equivalence relation R on X, there is a unique partition P with the property that x and y belong to the same cell of P if and only if xRy. Conversely, any partition P has associated with it a unique equivalence relation R constructed in the exact same manner: xRy if and only if x and y belong same cell within P. We give the cells of P a special name to celebrate its closeness with equivalence relations, we call the cells of P "equivalence classes." For each $x \in X$, we denote the equivalence class it belongs to by $[x]$. Note that $[x]$ is itself a set and it is unique. This caries the implication that given two equivalence classes $[x], [y]$, we have one of two cases. $[x] \cap [y] = [x]$ or $[x] \cap [y] = \emptyset$.

There is a highly important relation that we may discuss. It is of paramount importance when discussing topics in mathematical analysis. This is the function relation. xFy for $x \in X$ and $y \in Y$, for EACH x, there is a UNIQUE y for which $(x,y) \in F$. Since y is unique and depends on our choice of x, we write $y = f(x)$. We say that x MAPS TO y or y is the

image of x under f or y **corresponds** to x under f.
Given sets X and Y, we write $f : X \to Y$, to indicate the function takes in elements from X as inputs and outputs elements belonging to Y. We say that X is the domain of f and subset of Y for consisting of the elements that are images of some $x \in X$ is said to be the range of f. In the case in which the range of f is a a set of real numbers, namely $f : X \to \mathbb{R}^n$, we say f is a real-valued function. Similarly, if the range is complex, $f : X \to \mathbb{C}^n$, we say f is complex valued.
We denote the domain of f as D_f and the range of f as R_f. If $R_f = Y$, then we say that f is surjective or onto. Intuitively, this means that every element of Y is hit by some element in X. We may also say that f is injective or one-to-one, if whenever $f(x) = f(w)$ for some $f(x), f(w) \in R_f$, implies that $x = w$ for $w, x \in D_f$ Intuitively, this means that each element in D_f is mapped to a unique element in R_f. If f is both injective and surjective, we say f is bijective.
If $f : A \to B$ is bijective, then we know that each element in B was mapped by a unique element in A. Hence we may construct $g : Y \to x$ as the following relation, whenever $(x, y) \in F$, we have $(y, x) \in G$. We call this function the inverse of f. I then construct the idea of a composition to make use of the definition of an inverse. Let X, Y, Z be sets and let $f : X \to Y$ and $g : Y \to Z$. Define $h : X \to Z$ as $h = g \circ f$, with the property that xHz if $img_f(x) = pre_g(z)$, where $img_f(x)$ is the image of x under f and $pre_g(z)$ is the preimage of z under g.
If we take the inverse of f, which we denote as f^{-1}, where $f : A \to B$, more specifically $f : x \mapsto y$.