On Order

Matthew Hanna

May 6, 2019

Let S be a set. We may define an order relation which is a map $O : S \times S \to \{T, F\}$, where we write $O : (x, y) \mapsto T$ as $x < y$. There are rules for how this function operates.

1. O is somewhat anti symmetric
   - If $O(x, y) = T$, then $O(y, x) = F$
   - If $O(x, y) = F$ and $O(y, x) = F$, then we may define a new function E that has the exact same domain and range, that is, $E : S \times S \to \{T, F\}$. It sends (x, y) to T if and only if $O(x, y) = F$ and $O(y, x) = F$. We write this as $x = y$.
   - The previous 2 bullet points yield a trichotamy between $x < y$, $y < x$, and $x = y$.
2. Elements of S under the order O play nice with each other.
   - If $x < y$ and $y < z$, then $x < z$.
   - If $x = y$ and $y < z$, then $x < z$.
   - If $x < y$ and $y = z$, then $x < z$.
3. If $x < y$ or $x = y$, then we write $x \leq y$.

We define an order on $\mathbb{Q}$ as $x < y$ if $y - x \in \mathbb{Q}^+$
Let S be a set and let $E \subset S$. We say that E is bounded above if there exists $\beta \in S$ such for all $x \in E$, $x \leq \beta$. We call β an upper bound.
Similarly, if there exists $\beta \in S$ such for all $x \in E$, $x \geq \beta$. We call β a lower bound and say that E is bounded below.

Now suppose S is a set and that $E \subset S$ is bounded above. Now suppose there exists some $\alpha \in S$ with the following properties.

- α is an upper bound for E.
- If $\gamma < \alpha$, then γ is not an upper bound of E.

We call α the least upper bound of E. Note the use of the word "the" for if there were two distinct least upper bounds, name them α_1 and α_2 then either $\alpha_1 < \alpha_2$ or $\alpha_2 < \alpha_1$. In either case, the 2^{nd} condition takes effect and eliminates the smaller of the candidates. We write $\alpha = sup(E)$.
Similarly, if $E \subset S$ is bounded below and if there exists some $\alpha \in S$ with the following properties.

- α is a lower bound for E.
- If $\gamma > \alpha$, then γ is not a lower bound of E.

We say that α is the greatest lower bound for E. A similar argument as above tells us that the greatest lower bound is also unique.

Examples:

1. Consider $A = \{x \in \mathbb{Q} \mid x^2 < 2\}$ and $B = \{x \in \mathbb{Q} \mid x^2 > 2\}$. A pair of sets whose properties we have already extensively studied already. We know that A is bounded above by construction. And by the trichotimous property of the order relation and by the first theorem proved in this section, we know that B is precisely the set of upper bounds for A. The second theorem which tells that B has no smallest element tells us that A has no least upper bound. Because A did, call it p. Then, $p \in B$ and pick $q \in B$ for which $q < p$ whose existence is guaranteed by the second theorem. But when we have $q < p$, then q is not an upper bound for A. But since B is the collection of upper bounds for A, this means that $q \notin B$. We have a contradiction and hence A has no least upper bound in $\mathbb{Q}$.
   Similarly, A is the collection of lower bounds for B and a similar argument shows that B has no greatest lower bound in $\mathbb{Q}$.

2. Provided that $\alpha = sup(E)$ exists, it follows that α may or may not be an element of E. Consider $E_1 = \{x \in \mathbb{Q} \mid x < 0\}$ and $E_2 = \{x \in \mathbb{Q} \mid x \leq 0\}$. By definition, 0 is an upper bound for E_1 and E_2. If $x < 0$, then $x \in E_1 \cap E_2$. Hence, 0 is the least upper bound for both E_1 and E_2. But notice that $0 \notin E_1$ but $0 \in E_2$.

3. Consider $E = \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Since for all $x \in E$, $x \leq 1$, we have that 1 is an upper bound for E. And since $1 \in E$, it follows that $1 = sup(E)$. Since for all $x \in E$, $0 < x$, we have that 0 is a lower bound for E. Suppose that $x > 0$ is also a lower bound. Then,

   (a) For all $n \in \mathbb{N}$, $x \leq \frac{1}{n}$, since x is defined to be the lower bound for E.

   (b) Then, for all $n \in \mathbb{N}$, $\frac{1}{x} \geq n$ based on how inverses behave with the order relation on $\mathbb{Q}$. (Note: I have not proved this property but this is intuitive.)

   But by the Archimedean property, (b) is never true. Hence, x is not a lower bound for E. Therefore, $0 = inf(E)$ Note that, $1 \in E$ and $0 \notin E$

An ordered set S is said to have the least upper bound property if $E \subset S$ with $E \neq \emptyset$ and E is bound above implies that $sup(E) \in S$.

Theorem 1. *Suppose S is an ordered set with the least upper bound property. Suppose $B \subset S$, $B \neq \emptyset$ and B is bounded below. Let L be the set of all lower bounds of B. Then*

$$\alpha = sup(L)$$

exists in S and $\alpha = inf(B)$
In particular, $inf(B)$ exists in S.

Proof. Since B is non-empty and bounded below, we know that L is also non-empty. Moreover, for all $b \in B$ and for all $l \in L$, we have that $l \leq b$, since L is the set of lower bounds for B. This means that L is bounded above by elements of B and since S has the least upper bound property, we know that $\alpha = sup(L)$ exists in S.
Consider some $\gamma \in S$ for which $\gamma < \alpha$. Since α is defined to the supremum of L γ is not an upper bound for L. Since L is bounded above by all elements of B, it follows that $\gamma \notin B$.
Therefore, if $x \in B$, $\alpha \leq x$, by the reasoning that $(\gamma < \alpha) \to (\gamma \notin B)$. And since L is defined to be the set of lower bounds for B, it follows that $\alpha \in L$.
If $\alpha < \beta$, then $\beta \notin L$ since α is an upper bound.
Since L is the set of lower bounds, $alpha \in A$, and if $\alpha < \beta$, then $\beta \notin L$, then $\alpha = inf(B)$. □